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Robust optimum design of tuned mass dampers devices in random vibrations mitigation

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Abstract

One of the most widely adopted and studied strategies for vibration control both in civil and in mechanical engineering is based on the use of tuned mass dampers (TMD) devices. Many conventional optimization criteria of mechanical parameters have been proposed, based on different approaches typically of a ‘‘conventional’’ type; in other words, they are based on the implicit assumption that all parameters involved are deterministically known. Removing this hypothesis means to convert a conventional optimization into a robust one, so that the solution must be able not only to minimize a performance but also to limit its variation induced by uncertainty in system parameters. In this work, a robust optimal design criterion for a single TMD device is proposed. The analyzed case concerns the structural vibration control of a main system subject to stochastic dynamic loads by a single linear TMD. The dynamic input is represented by a random base acceleration, modelled by a stationary filtered white noise process. It is assumed that not only mechanical parameters regarding main structure and TMD but also input spectral contents are affected by uncertainty. The problem is treated characterizing all uncertain parameters by a nominal mean value and a variance. It is also assumed that all these parameters are statistically independent. The protected main structure covariance displacement (dimensionless by dividing for the unprotected one) is adopted as the deterministic objective function (OF). Its mean and standard deviation are evaluated to perform the robust design. Robustness is formulated as a multiobjective optimization problem, in which both the mean and the standard deviations of the deterministic OF are minimized. Comparisons with a conventional approach based on the same OF show that the robust approach induces a significant improvement in performance stability.

1. Introduction

An important task in the field of structural optimization is the response evaluation when dispersion of system parameter values is considered. Uncertainty of structural problems could attain many elements, which are considered deterministic in standard structural analysis, such as loads intensity or mechanical and

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Nomenclature

A_S	structural system matrix
$\mathbf{A}_{,d_i}$	derivative of the state matrix $\mathbf{A}$
$\mathbf{A}_{0,d_i}$	derivative of the state matrix $\mathbf{A}_0$
$\mathbf{A}_0$	state matrix of the unprotected structure
$\bar{b}$	design vector
B_0	matrix of the Lyapunof equation for the unprotected structure
$\mathbf{B}_{,d_i}$	derivative of matrix $\mathbf{B}$
$\mathbf{B}^0_{,d_i}$	derivative of matrix $\mathbf{B}_0$
$\mathbf{B}$	matrix of the Lyapunov equation for the protected structure
C	damping system matrix
$\bar{d}$	vector collecting the uncertain parameters
K	stiffness system matrix
M	mass system matrix
$\mathbf{R}_{ZZ}$	state space covariance matrix
$\mathbf{R}_{Z_0Z_0}$	unprotected main structural state space covariance matrix
S_0	power spectral density intensity of white excitation at the bed rock
T_f	filter period
T_s	main system period
$w(t)$	stationary Gaussian zero mean white noise process
$\ddot{y}_b(t)$	acceleration that excites the system at the base
y_s	relative displacement of the main structure
y_T	relative displacement of the TMD
$Y_f(t)$	displacement process of the filter
$\ddot{\bar{Y}}(t)$	relative base acceleration vector
$\bar{Y}(t)$	relative base displacement vector
$\dot{\bar{Y}}(t)$	relative base velocity vector
$\ddot{\bar{Y}}_b(t)$	stationary input base acceleration
$\ddot{Y}_f(t)$	relative acceleration process of the filter
$\dot{\bar{Y}}_f(t)$	velocity process of the filter
$\bar{Z}_s(t)$	state space vector of the main structure-TMD system

Greek letters

β_i	sensitivity coefficient
μ_x	nominal mean value of the variable x
η	ratio between the TMD and main structure masses
ρ_x	correlation of the variable x
σ_{X_S}	standard deviation of maximum displacement of the protected structure
$\sigma^0_{X_S}$	standard deviation of maximum displacement of the unprotected structure
ξ_f	filter damping
ξ_s	damping of the main structure
ξ_T	damping of the TMD
ρ_x	variation coefficient of the parameter x
Ψ	frequency ratio
ω_f	base filter frequency
ω_s	main structural circular frequency
ω_T	tuned mass damper circular frequency
Ω	admissible domain of the design parameters

geometrical configurations. A typically simplified approach is that where the only source of randomness is assumed to be dynamic load with a stochastic nature, as in case of earthquake or wind actions. The actions can be modelled by a stochastic process and the standard random vibration theory can be used [1] when all the other motion equation parameters are considered deterministic. This approach gives structural response characterization completely described by stochastic processes with deterministic parameters. Under these assumptions, a "conventional" stochastic structural optimization (CSSO) could be performed in the sense that except for load nature, all the other parameters involved are assumed as unaffected by any uncertainty source. In the field of random vibration, the first definition of structural optimal problem was proposed by Nigam [2], leading to a standard nonlinear restraint problem in which restraints were defined by probabilistic structural response indexes while the objective function (OF) by structural weight. The use of a defined stochastic OF was later proposed for optimal damping design in the domain of seismic protection [3]. The optimal damping value of a device placed on the first story of a building was determined by minimizing an OF defined by the maximum structural displacement under white noise excitation. More recently, a specific and more complete stochastic approach has been proposed by Takewaki [4], in order to obtain a stiffness-damping simultaneous optimization of structural systems. The sum of mean square responses due to a stationary random excitation is minimized under constraints on total stiffness capacity and total damper capacity. An interesting alternative stochastic approach for damping devices optimum design has been proposed by Kwan-Soon et al. [5] to minimize the total building life-cycle cost. It is based on a stochastic dynamic approach for failure probability evaluation, while the OF is defined in a deterministic way. The conventional stochastic optimization problem is also formulated by adopting the location and the amount of the viscous-elastic dampers as design variables [6]. Constraints are the maximum inter-storey drifts evaluated by the first crossing theory application in non-stationary conditions. Another interesting work regards unrestricted optimization of single nonlinear [7] and multiple linear [8] tuned mass dampers (TMD) by using as OF the structural displacement covariance of the protected system where input is treated by a simple stationary white noise. A complete stochastically defined CSSO is proposed by Marano et al. [9]. In this latter work, a based-optimum criterion is developed by adopting a covariance reliability approach. Both the OF and constraints are defined in a stochastic way. In detail constraints impose a limit to failure probability associated with the first threshold crossing of structural displacement.

It is rather clear that the implicit assumption that uncertainties in a structural system have negligible effects on response is a further simplification in many real situations. However, it was reported that the uncertainty in structural parameters might have equal or even greater influence on the response than the uncertainty in excitations [10]. This could be particularly significant for those cases where solution is strongly influenced by system parameters variation, as for structural optimization. For example, design of civil structures in seismically active regions requires consideration of both the uncertainty in e arthquake ground motions and the uncertainty of design-base structural models. This problem is challenging and only a limited number of publications deals with both uncertainties, for example, Refs. [11–17].

The treatment of uncertainty in engineering and in structural design is still an open question and the scientific literature offers different approaches based on dissimilar mathematical models. The probabilistic technique, here used, most common one, not only for confidence that engineers have with this approach, but also because different ways have been proposed and applied, as fuzzy and interval analysis, just to cite a few (for example, see Ref. [18]). But the main point in this field is that there is no mathematical and engineering certainly on the best way to have knowledge of uncertainty. Typically, the probabilistic approach is the most complete, being its information more detailed in comparison with the other methodologies. But the main engineering difficulty is in obtaining enough information to be confident on the adopted probabilistic model. Nevertheless, the selection of a specific probability density function is a hard problem. In these cases, alternative approaches may be used to overcome this limitation.

Thus, for a more realistic analysis, system parameters must be treated by a suitable description of uncertainty that afflicts their nominal values. Because of different uncertain factors in materials, measurement, manufacturing and installment, practical structures in mechanical or civil engineering are often more realistically described by random variables. For the same reason also safe domain and input process parameters, as power spectral density, have to be treated as uncertain quantities.

This means that CSSO may not achieve or may be infeasible due to the scatter of structural behaviour. Therefore, it is reasonable to explore the effects of uncertainty on the design of structures subject to random vibrations. For this reason, in the last 20 years, various non-deterministic methods have been developed in order to deal with optimum design under environmental uncertainties.

These methods can be classified into two main approaches, namely *reliability-based methods* and *robust design-based methods*:

- *Reliability methods* estimate the probability distribution of the system's response based on the known probability distribution of the random parameters. They are mostly used for risk analysis by computing the probability of the failure. However, the variation is not minimized in the reliability approach [19], which concentrates on the rare events at the tails of probability distribution [20].
- *Structural robust design* (SRD) optimizes a performance index in terms of mean value, and at the same time it minimizes its variability resulting from environmental uncertainty. The final solution is less sensitive to the parameters variation, while maintaining eventually feasibility with probabilistic constraints. This is achieved by optimizing the design vector (DV) in order to make the performance minimally sensitive to the different causes of variation.

Hence, robust design concentrates on the probability distribution close to the mean value. Thus an SRD solution is not able to give the best performance in an absolute sense but it provides a lower sensitivity to uncertainty. Recently a robust design of a vibration absorber, with mass and stiffness uncertainty in the main system, is used to demonstrate the robust design approach in dynamic as proposed by Zang et al. [21]. It is based on a frequency approach and it assumes that input is a band-limited white noise. Uncertainty is defined by mean and covariance, and it concerns with main system mass and damping. As local performance index is used the maximum over a limited frequency band of the dimensionless displacement transfer function, and the robust optimization has been obtained by minimizing its deviation in mean and variance. This optimization is obtained by a direct first-order perturbation method based on a Taylor-series expansion.

In this paper, the practicable applicability of the proposed robust optimization is shown by means of a single TMD device on a single-degree-of-freedom (sdof) system in order to evaluate the global effectiveness of the method, under earthquake excitation. Even if the structural analysis of a general N-dof system protected by a TMD device must be placed as an $N+1$ degree of freedom system mechanical model to be carried out in the dimensional space under general conditions, let us observe that a single TMD can only be tuned to a single structural frequency. Therefore, it is expected that its effectiveness is the greatest if an N degrees of freedom structural system oscillates around a predominant mode [22].

Therefore it is quite common considering only one main structural vibration mode, typically the first one, as descriptive of the protected element vibration for the TMD optimal design.

The TMD is introduced in order to guarantee a suitable protection level in the primary structure for both the structure and its contents, towards a defined limit state. Its mechanism of attenuating detrimental vibrations of a structure is to transfer the vibration energy of the structure to the TMD, and to dissipate the energy through the damping of the TMD. A multiobjective approach for robust design of TMD system is presented. The structural configuration is represented by a primary viscous-elastic system subject to a base acceleration. This last is modelled by a stationary filtered white noise stochastic process. The primary element is protected by a linear single TMD against vibration induced by base acceleration. It is assumed that uncertainty in problem parameters deals with primary system frequency and damping and base excitation spectral contents. The DV here used contains the TMD frequency and damping, and the ratio between the protected and the unprotected standard deviation of the main system displacements is used as OF. The robust solution is obtained by multiobjective criteria where optimization is achieved by minimizing both the OF mean value and standard the deviation. Indeed, it is demonstrated that the design optimizes the structural performance if also minimizes its robustness to uncertainty. Mean and covariance OFs are here obtained by applying the direct perturbation method (DPM) first-order approximation. A non-dominated sorting in genetic algorithm (GA) in its second version (NSGA-II) (a Pareto-based multiobjective evolutionary algorithm) is used in order to evaluate the set of possible solutions. Finally, a comparison between conventional deterministic and robust stochastic optimal solutions is shown.

2. Linear elastic TMD system subject to stationary base random vibration

The concept of passive structural control is widely accepted and it has been frequently applied to civil and mechanical structures. Among the numerous passive control methods available, the TMD is one of the simplest and one of the most reliable control devices, not only for new structures but also for existing ones. The principle of vibration absorber was attributed to Frahm [23] who found that a natural frequency of a structure could be split into two frequencies by attaching a small spring–mass system tuned to the same frequency as the structure. It is composed of mass, a spring and a damper (Fig. 1), and is widely used as a passive control device in practical application for suppressing the undesirable vibration mainly due to its simplicity and high reliability. Its mechanism of attenuating undesirable vibration of a structure is to transfer the vibration energy of the structure to the TMD and to dissipate the energy through the damping of the TMD. In civil engineering applications, the oscillator could be a high-rise building, bridge or offshore platform. The use of a TMD would be intended to reduce wind, earthquake or wave-induced vibrations.

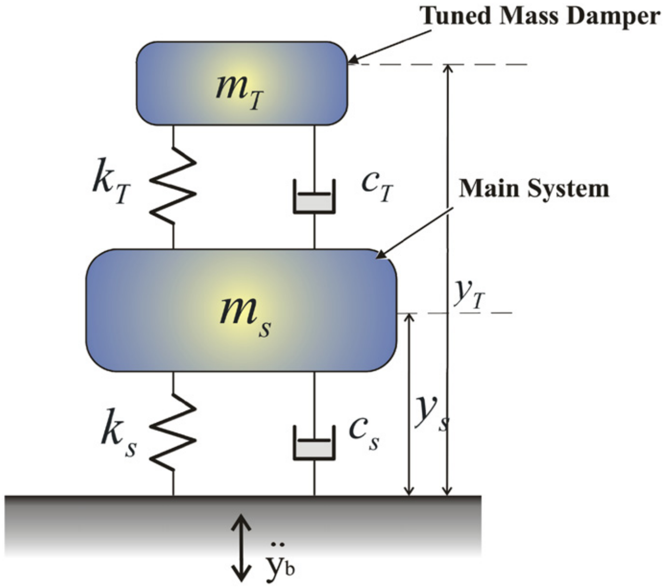

Fig. 1. Linear scheme of TMD system.

In case of a TMD system excited by a base acceleration, the structural response is determined by solving the dynamic equilibrium system equations

$$\mathbf{M}\ddot{\bar{Y}}(t) + \mathbf{C}\dot{\bar{Y}}(t) + \mathbf{K}\bar{Y}(t) = \bar{r}\ddot{y}_b(t), \tag{1}$$

where $\bar{Y} = (y_S, y_T)^{\mathrm{T}}$ is the relative base displacement vector, and $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass, damping and stiffness symmetric matrices.

Introducing the state space vector

$$\bar{Z}_s = (y_T, y_S, \dot{y}_T, \dot{y}_S)^{\mathrm{T}}, \tag{2}$$

system could be replaced by

$$\dot{\bar{Z}}_s(t) = \mathbf{A}_S\bar{Z}_s(t) + \bar{r}_z\ddot{y}_b(t), \tag{3}$$

where

$$\mathbf{A}_s = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{H}_s^1 & \mathbf{H}_s^2 \end{pmatrix} \tag{4}$$

is the structural system matrix, $\bar{r}_z = (0,0,1,1)^{\mathrm{T}}$, $\mathbf{I}$ and $\mathbf{0}$ the unit and zero 2×2 matrices, respectively, and

$$\mathbf{H}_s^1 = \mathbf{M}^{-1}\mathbf{K} = \begin{pmatrix} -\omega_T^2 & +\omega_T^2 \\ +\eta\omega_T^2 & -(\eta\omega_T^2 + \omega_S^2) \end{pmatrix}, \tag{5}$$

$$\mathbf{H}_s^2 = \mathbf{M}^{-1}\mathbf{C} = \begin{pmatrix} -2\xi_T\omega_T & +2\xi_T\omega_T \\ +\eta 2\xi_T\omega_T & -(\eta 2\xi_T\omega_T + 2\xi_S\omega_S) \end{pmatrix}, \tag{6}$$

where the system mechanical parameters are

$$\omega_T = \sqrt{\frac{k_T}{m_T}}, \tag{7}$$

$$\omega_S = \sqrt{\frac{k_S}{m_S}}, \tag{8}$$

$$\xi_T = \frac{c_T}{2\sqrt{m_T k_T}}, \tag{9}$$

$$\xi_S = \frac{c_S}{2\sqrt{m_S k_S}}, \tag{10}$$

$$\eta = \frac{m_T}{m_S}. \tag{11}$$

The load $\ddot{y}_b(t)$ represents the acceleration that excites the system at the base. Due to the fact that it can be modelled as a stochastic process, many advantages could be reached if it is filtered by a white noise.

For base random accelerations modelling, a widely adopted model in both stationary and non-stationary cases is that obtained by a simple linear second-order filtering of the white noise process. It is able to characterize input frequency modulation for a wide range of practical situations, and in case of non-stationary input it is able to model not only amplitude but also frequency contents time variation. For the general case of stationary input base acceleration, $\ddot{\bar{Y}}_b(t)$ is expressed as

$$\begin{cases} \ddot{\bar{Y}}_f(t) + 2\xi_f\omega_f\dot{\bar{Y}}_f + \omega_f^2\bar{Y}_f = -w(t), \\ \ddot{\bar{Y}}_b(t) = \ddot{\bar{Y}}_f(t) + w(t) = -(2\xi_f\omega_f\dot{\bar{Y}}_f + \omega_f^2\bar{Y}_f), \end{cases} \tag{12}$$

where $w(t)$ is a stationary Gaussian zero mean white noise process whose intensity is given by S_0,[1] representing the excitation at the bed rock, ω_f is the base filter frequency and ξ_f the filter damping.

The global state space vector is

$$\bar{Z} = (y_T \quad y_S \quad y_f \quad \dot{y}_T \quad \dot{y}_S \quad \dot{y}_f)^{\mathrm{T}}. \tag{13}$$

The state space covariance matrix $\mathbf{R}_{ZZ}$ is then obtained as solution of the *Lyapunov* equation, which in this case is represented by a 6×6 order algebraic matrix equation:

$$\mathbf{A}\mathbf{R}_{ZZ} + \mathbf{R}_{ZZ}\mathbf{A}^{\mathrm{T}} + \mathbf{B} = \mathbf{0}, \tag{14}$$

where

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\omega_T^2 & +\omega_T^2 & +\omega_f^2 & -2\xi_T\omega_T & +2\xi_T\omega_T & +2\xi_f\omega_f \\ +\eta\omega_T^2 & -(\eta\omega_T^2+\omega_S^2) & +\omega_f^2 & +\eta 2\xi_T\omega_T & -(\eta 2\xi_T\omega_T + 2\xi_S\omega_S) & +2\xi_f\omega_f \\ 0 & 0 & -\omega_f^2 & 0 & 0 & -2\xi_f\omega_f \end{pmatrix}. \tag{15}$$

And the 6×6 matrix $\mathbf{B}$ has all null elements except for the last one on the main diagonal:

$$[\mathbf{B}]_{6,6} = 2\pi S_0. \tag{16}$$

The unprotected main structural response in covariance 4×4 matrix $\mathbf{R}_{Z_0Z_0}$ is valuable in the same way:

$$\mathbf{A}_0\mathbf{R}_{Z_0Z_0} + \mathbf{R}_{Z_0Z_0}\mathbf{A}_0^{\mathrm{T}} + \mathbf{B}_0 = \mathbf{0}, \tag{17}$$

where the state space vector and system matrix are now

$$\bar{Z}_0 = \left\{ y_S \quad y_f \quad \dot{y}_S \quad \dot{y}_f \right\}^{\mathrm{T}}, \tag{18}$$

[1] $E[w(t)w(t-\tau)] = 2\pi S_0\delta(t-\tau)$.

$$\mathbf{A}_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\omega_S^2 & \omega_f^2 & -2\xi_S\omega_S & 2\xi_f\omega_f \\ 0 & -\omega_S^2 & 0 & -2\xi_f\omega_f \end{pmatrix} \tag{19}$$

and finally the 4×4 matrix $\mathbf{B}_0$ has all null elements except for the last one on the main diagonal that is

$$[\mathbf{B}]_{4,4} = 2\pi S_0. \tag{20}$$

Moreover, previous formulation deals with the assumption that all structural parameters have a deterministic nature but this usually is an unrealistic assumption if referred to many real cases.

In this work, three *structural parameters* are modelled as afflicted by uncertainty:

- the main structural circular frequency ω_S,
- the main structural damping ξ_s,
- the ratio between the TMD and the main structure masses η.

The first one is often quite difficult to predict accurately. The actual values are usually determined by full-scale measurements after the structure is constructed and may vary with time. For this reason, it is desirable that the natural frequency of the TMD be tunable on site [24]. The main system damping, as known, has a very limited influence on optimal TMD parameters, as observed by different authors [6,25], and then a little uncertainty has generally been assumed. Nevertheless, there is almost uncertainty about main system energy dissipation during the dynamic motion of mechanical systems and then the components of damping matrix are typically afflicted by inconfident evaluation. Finally, also the mass of the main system may be affected by significant variations during the service life of the mechanical systems, especially if it represents a civil structure such as buildings and bridges.

3. Probabilistic characterization of uncertain system parameters

The most common approach in modelling of the uncertain structural parameters, is using probability analysis so that each uncertain parameter is treated as a random variable characterized by standard distribution. This means that the problem must be solved by using a multidimensional *Joint Probability Density* Function of all the involved parameters. Nevertheless, this way often offers serious analytical and numerical difficulties. Moreover, it also presents some conceptual limitations because the complete uncertain parameters stochastic characterization presents a fundamental limitation that is related to the difficulty of a complex statistical analysis that cannot be justified in the common real situation characterized by the absence of detailed statistical input data. A simplification is usually given by assuming that all variables have independent normal or lognormal distributions as application of limit central theorem, but this way does not overcome the previous problem. On the other side, it is quite usual to approach real situations where it is only possible to estimate the mean and variance of each uncertain parameters being not possible to have more information about their real probabilistic distribution. Then, this specific case is treated assuming that all uncertain parameters that are collected in vector $\bar{d}$ are characterized by a nominal mean value μ_{d_i}, and a correlation

$$\rho_{d_i} = \frac{\sigma_{d_i}}{\mu_{d_i}}. \tag{21}$$

Structural parameters assumed as uncertain are the main system frequency ω_S and damping ξ_S and the mass ratio η.

Moreover, also the two *filter parameters* ω_f and ξ_f are assumed uncertain, so that both of them are characterized by a variance more than nominal value.

Therefore, the uncertain parameters vector $\bar{d}$ is composed by the following elements:

$$\bar{d} = (\omega_S, \xi_S, \eta, \omega_f, \xi_f). \tag{22}$$

For the sake of simplicity in the remaining part of the paper, the mean value of each uncertain element will be simply indicated by its nominal symbol:

$$\mu_{d_i} = d_i.$$

4. Conventional and robust optimum design of TMD subject to random vibration

The optimization problem for a structure subject to random vibrations could be formulated as the search of a suitable set of variables (that are the parameters of the design characterizing structural configurations), collected in the so-called *DV* $\bar{b}$, over a possible admissible domain $\boldsymbol{\Omega}$. The optimal DV must be able to reduce the vibration induced under an acceptable level, minimizing a given OF (defined by using deterministic or statistic entities) and also satisfying particular constraints, expressed in terms of structural reliability. Both reliability constraints and OF must be defined over a given time interval, as the problem regards dynamic structural response.

Two possible approaches can be performed to solve the structural optimization problem: one of *conventional* type, in which only the loads are considered affected by uncertainty, or a *robust* optimization, based on the assumption that also the system parameters are uncertain.

4.1. The conventional CSSO

The conventional optimization problem so defined and first stated by Nigam for a system subject to random vibrations [2], can be transformed into a standard nonlinear programme that is stated as

$$\text{find} \quad \bar{b} \in \boldsymbol{\Omega}_b, \tag{23}$$

$$\text{that minimize} \quad \text{OF}(\bar{b}, t), \tag{24}$$

$$\text{subject to} \quad g_i(\bar{b}, t) \leqslant 0 \quad (i = 1, 2, \ldots, k), \tag{25}$$

where the OF could be defined by a standard deterministic way (for example total structural weight or elements volume) or in a stochastic one. In this last case, statistic entities could be used as covariance or spectral moments of variables of interest (for example displacement, acceleration or structural stress in relevant elements). Also, restraints could regard spectral or statistical moment or, in a more realistic way, reliability limitations, as for example in the following form:

$$P_f(\bar{b}, T) - P_f^{\text{adm}} \leqslant 0, \tag{26}$$

where P_f^{adm} is the maximum admissible failure probability that could be accepted by designers, and $P_f(\bar{b}, T)$ is the evaluated structural probability of failure during the lifetime [0,*T*].

In this specific case, the optimal mechanical parameters of a TMD are represented by the two-dimensional DV $\bar{b}$:

$$\bar{b} = (\omega_T, \xi_T)^{\text{T}}, \tag{27}$$

having assigned the fixed mass ratio η and the main system frequency ω_S.

It is determined by minimizing the variance of main mass displacement with respect to the base.

The OF is thus defined in a dimensionless way as the ratio between the standard deviation of maximum displacement of the protected structure σ_{X_S} and the unprotected one $\sigma^0_{X_S}$:

$$\text{OF} = \frac{\sigma_{X_S}}{\sigma^0_{X_S}}. \tag{28}$$

This function represents a direct stochastic *index of vibration protection effectiveness* that shows protection effectiveness when its value is smaller than one. At the same time, a value of the OF close to the unit indicates practically negligible effects in vibration control (greater values are for negative TMD effects, increasing main structure displacements). The CSSO is performed assuming that all parameters involved in the problem are

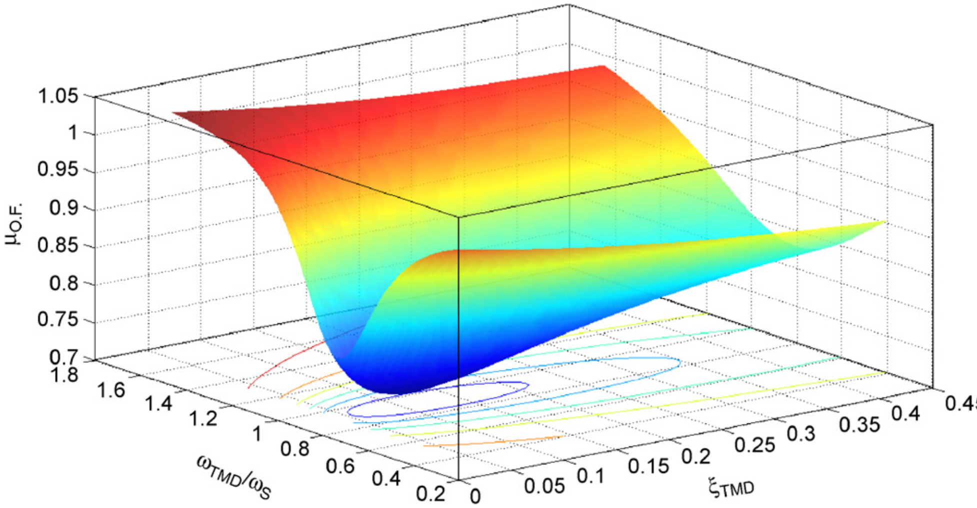

Fig. 2. Deterministic OF surface in the dimensionless DV space ($\omega_{\rm TMD}/\omega_S$) and $\xi_{\rm TMD}$. Forcing input is characterized by ξ_f and ω_f. Main system parameters are circular frequency ω_S, damping ratio ξ_S and mass damping η.

deterministic, and the unrestricted approach of Eqs. (23) and (24), is for this specific case:

$$\text{find} \quad \bar{b} = (\omega_T, \xi_T) \in \Re_+^2, \tag{29}$$

$$\text{that minimize} \quad \frac{\sigma_{X_S}(\bar{b})}{\sigma^0_{X_S}}. \tag{30}$$

This approach is able to furnish a global minimum value, as can be seen in Fig. 2.

4.2. The robust structural optimization

As stated before, for design of structures with parameters stochastically defined and subject to random dynamic loads, one possible approach is to define the optimal conditions, as the mean value, of that corresponding deterministic parameter conditions. However, the optimal solution obtained by minimizing the expected value of the OF, defined in Eq. (24), may be quite sensitive to the fluctuation of stochastic parameters, causing scatter of the performance. Thus, a more robust design concept has to be adopted to overcome this limitation. In the proposed case of an unconstraints optimization problem, a larger deviation of structural performance from its usually expected value takes place. A solution that could be defined as "robust" is that which characterizes completely the OF as a random variable and then it can be completely described by mean of the knowledge of its probability density function. Nevertheless, this way presents many difficulties and it can be found in an analytical form only in few cases. To overcome this limitation, the complete OF statistical description is replaced by the knowledge of its first two statistic moments, the mean value and the variance. One of the simplest approaches for calculating the effects of the uncertainties parameters is the *DPM*, which consists in approximating the response as a polynomial of the uncertain parameters (see, for example, Ref. [1]). More precisely, the polynomial is a Taylor series about the nominal mean values of the uncertain parameters. Only the knowledge of mean and covariance of uncertain parameters is required.

If $R(\bar{d})$ denotes the generic stochastic structural response, which depends on the uncertain parameter vector $\bar{d}$ (displacement, velocity, acceleration, reliability, etc.), the linear approximation of the DPM furnishes its mean value and variance as described in the following:

$$\mu_{[R]_{\rm lin}} = R(\mu_{\bar{d}}), \tag{31}$$

$$\sigma_{R_{\text{lin}}} = \sum_{i=1}^{n_d}\sum_{j=1}^{n_d}(\beta_i\beta_j)\,\text{cov}[d_i d_j], \tag{32}$$

where n_d is the dimension of the uncertain element vector, and $\beta_i = (\partial R/\partial d_i)_{\bar{d}=\bar{\mu}_D}$ are the sensitivity coefficients evaluated for the mean value of vector $\bar{d}$. This last formulation takes considerable simplifications if vector $\bar{d}$ components are assumed as statistically independent (and therefore uncorrelated). Eq. (32) becomes, in this specific case, as follows:

$$\sigma_{R_{\text{lin}}} = \sqrt{\sum_{i=1}^{n_d}\beta_i^2\sigma_{d_i}^2}. \tag{33}$$

With reference to the OF defined in Eqs. (30), Eqs. (31) and (32) become

$$\mu_{\text{OF}}(\bar{d},\bar{b}) = \text{OF}(\mu_{\bar{d}},\mu_{\bar{b}}), \tag{34}$$

$$\sigma_{\text{OF}}(\bar{b},\bar{d}) = \sqrt{\sum_{i=1}^{n_d}\left\{\left(\frac{\partial}{\partial d_i}\text{OF}(\bar{b},\bar{d})\right)^2_{\mu_{\bar{d}}}\sigma_{d_i}^2\right\}}, \tag{35}$$

where σ_{d_i} is problem data, and

$$\left(\frac{\partial}{\partial d_i}\text{OF}(\bar{b},\bar{d})\right)\Bigg|_{\mu_d} = \frac{(\sigma_{X_S})_{,d_i}\sigma_{X_S^0} - \sigma_{X_S}(\sigma_{X_S^0})_{,d_i}}{\sigma_{X_S^0}^2}, \tag{36}$$

where $(\bullet)_{,d_i} = (d\bullet/d(d_i))$.

The two terms in Eq. (36) directly obtainable from Eqs. (14) and (17) are

$$\sigma_{X_S} = \sqrt{[\mathbf{R}_{ZZ}]_{2,2}}, \tag{37}$$

$$\sigma_{X_S^0} = \sqrt{[\mathbf{R}_{Z_0Z_0}]_{1,1}}. \tag{38}$$

The other two quantities that are their first derivative are obtainable:

$$(\sigma_{X_s})_{,d_i} = \left(\frac{\text{d}\sigma_{X_s}(d,b)}{\text{d}(d_i)}\right) = \frac{1}{2}\frac{([\mathbf{R}_{ZZ}]_{22})_{,d_i}}{\sqrt{[\mathbf{R}_{ZZ}]_{22}}}, \tag{39}$$

$$(\sigma_{X_s^0})_{,d_i} = \left(\frac{d\sigma_{X_S^0}(d,b)}{d(d_i)}\right) = \frac{1}{2}\frac{([\mathbf{R}_{Z_0Z_0}]_{11})_{,d_i}}{\sqrt{[\mathbf{R}_{Z_0Z_0}]_{11}}}. \tag{40}$$

Both are obtained by deriving the original ones (14) and (17), so that it furnishes $\mathbf{R}_{,d_i}$:

$$\mathbf{A}\mathbf{R}_{ZZ,d_i} + \mathbf{R}_{ZZ,d_i}\mathbf{A}^{\text{T}} + \mathbf{C}_i = \mathbf{0}, \tag{41}$$

$$\mathbf{A}_0\mathbf{R}_{Z_0Z_0,d_i} + \mathbf{R}_{Z_0Z_0,d_i}\mathbf{A}_0^{\text{T}} + \mathbf{C}_i^0 = \mathbf{0}, \tag{42}$$

where

$$\mathbf{C}_i = \mathbf{A}_{,d_i}\mathbf{R}_{ZZ} + \mathbf{R}_{ZZ}\mathbf{A}_{,d_i}^{\text{T}} + \mathbf{B}_{,d_i}, \tag{43}$$

$$\mathbf{C}_i^0 = \mathbf{A}_{0,d_i}\mathbf{R}_{Z_0Z_0} + \mathbf{R}_{Z_0Z_0}\mathbf{A}_{0,d_i}^{\text{T}} + \mathbf{B}_{,d_i}^0, \tag{44}$$

where $\mathbf{A}_{,d_i}$ and $\mathbf{A}_{0,d_i}$ are the derivative of state matrices $\mathbf{A}$ and $\mathbf{A}_0$ with respect to each uncertain parameter. Moreover, both $\mathbf{B}_{,d_i}$ and $\mathbf{B}_{,d_i}^0$ are null matrices for all the vector $\bar{d}$ elements, so that both Equations (43) and (44) can be simplified. In this way, all quantities in equation relative to generic structural response are known, and it is possible to obtain the linear approximation of the OF mean value and variance in case of system parameters and frequency input content uncertainty. In this case, a possible way for the task of robust optimal structural design is to minimize dispersion of the OF by multicriteria measures of goal performance.

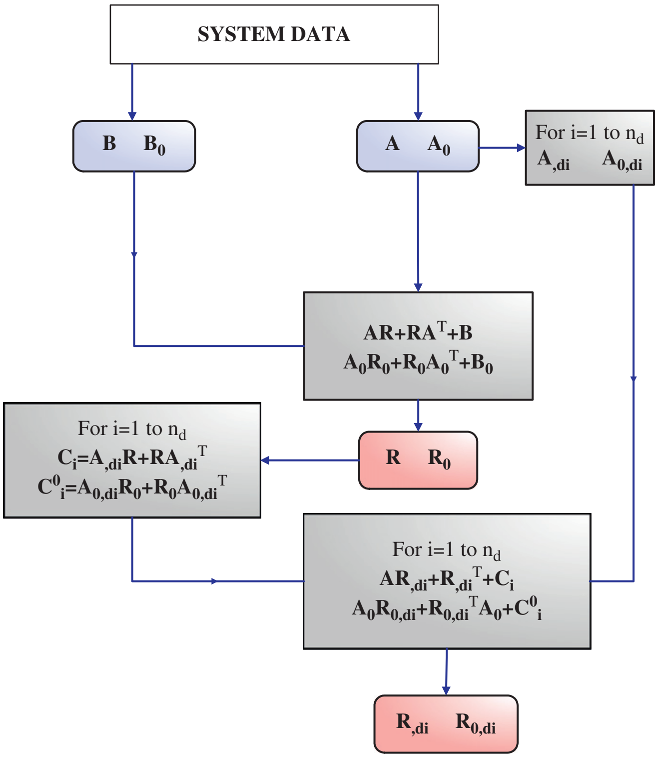

Fig. 3. Flow chart for evaluation of state space covariance matrices and its first derivatives, referred to the protected and unprotected configurations.

By adopting this formulation, the proposed problem becomes a vectorial minimization one, in which the two conflicting criteria are the mean value and the variance/standard deviation of the OF, that is:

$$\text{find} \quad \bar{b} \in \boldsymbol{\Omega}_b, \tag{45}$$

$$\text{that minimize} \quad \{\mu_{\mathrm{OF}}(\bar{b}), \sigma_{\mathrm{OF}}(\bar{b})\}. \tag{46}$$

The schematic flow chart to obtain covariance matrices and their first-order sensitivities of protected and unprotected structure is represented in Fig. 3.

5. Methods for multiobjective optimization problem (MOOP)

Many real engineering problems often involve several OFs in conflict with each other and for them it is not possible to define a universally approved criterion of ‘‘optimum’’ as in single objective optimization problem. Therefore, in MOOP the aim is to produce a set of good compromised solutions, among which the decision maker selects one [21]. Thus, optimization could be obtained by assuming that only one ‘‘effectiveness’’ index must be minimized, and that the others must be considered as problem constraints. Moreover, the definition of the best index to be minimized and the selection of the indices that must be transformed in constraints, has not a single definition. The above-mentioned question depends strongly on designer opinion and experiences. On the contrary, the multiobjective optimization gives to the designer the opportunity to evaluate a set of possible solutions, defined as those able to satisfy in the best way and with different performances all the required efficiency indices defined by designers. The definitions of these solutions are usually known as the *Pareto dominance* and *Pareto optimality* criterion that constitute a basic point in the MOOPs. With reference to the Pareto optimality definition, it assumes that a DV **b*** is Pareto optimal if no feasible vector **b** exists, which would decrease some criterion without causing a simultaneous increase in at least one other criterion. Unfortunately, this concept almost always gives not a single solution but rather a set of solutions called the

Pareto optimal set. The vectors $\mathbf{b}^*$ corresponding to the solutions included in the Pareto optimal set are called non-dominated. Generally, Pareto concepts ("*Pareto dominance*" and "*Pareto optimality*") constitute very important notions in MOOPs.

Without loss in generality, a typical minimization-based MOOP is assumed. Given two candidate solutions $\{\mathbf{d}_j, \mathbf{d}_k\}$, if

$$\forall i \in \{1,\ldots,M\}, \mathrm{OF}_i(d_j) \leqslant \mathrm{OF}_i(d_k) \wedge \exists i \in \{1,\ldots,M\} : \mathrm{OF}_i(d_j) < \mathrm{OF}_i(d_k) \tag{47}$$

defined the two objective vectors

$$\mathbf{v}(\mathbf{d}_j) = \{\mathrm{OF}_1(\mathbf{d}_j), \ldots, \mathrm{OF}_M(\mathbf{d}_j)\}, \tag{48}$$

$$\mathbf{v}(\mathbf{d}_k) = \{\mathrm{OF}_1(\mathbf{d}_k), \ldots, \mathrm{OF}_M(\mathbf{d}_k)\}, \tag{49}$$

vector $\mathbf{v}(\mathbf{d}_j)$ is said to dominate vector $\mathbf{v}(\mathbf{d}_k)$ (denoted by $\mathbf{v}(\mathbf{d}_j) \prec \mathbf{v}(\mathbf{d}_k)$).

Moreover, if no feasible solution ($\mathbf{v}(\mathbf{d}_k)$) exists that dominates solution $\mathbf{v}(\mathbf{d}_j)$, then $\mathbf{v}(\mathbf{d}_j)$ is classified as a *non-dominated* or *Pareto optimal solution*. In a more simple way, $\bar{\boldsymbol{b}}_j \in \boldsymbol{\Omega}_{\bar{b}}$ is a *Pareto optimal solution* if exists no feasible vector $\bar{\boldsymbol{b}}_k \in \boldsymbol{\Omega}_{\bar{b}}$, which would decrease some criterion without causing a simultaneous increase in at least one other criterion [26]. The collection of all Pareto optimal solutions are known as the *Pareto optimal set* or *Pareto efficient set*, instead the corresponding objective vectors are described as the *Pareto front* or *Trade-off surface*.

Normally, the decision about the "best solution" to be adopted is formulated by the so-called (human) *decision maker* (DM). Extremely rare is the case in which the DM does not have any role and a generic *Pareto optimal solution* is considered acceptable (*no-preference-based methods*). On the other hand, several *preference-based methods* exist in the literature, although this particular face of research tends to have been somewhat overlooked. A more general classification of the *preference-based method* considers when the preference information is used to influence the search [27]. Thus, in *a priori methods*, DM's preferences are incorporated before the search begins; therefore, based on the DM's preferences, it is possible to avoid to produce the whole *Pareto optimal set*. In *progressive methods*, the DM's preferences are incorporated during the search: this scheme offers the sure advantage of driving the search process but the DM may be unsure of his/her preferences at the beginning of the procedure and may be informed and influenced by information that becomes available during the search. A last class of methods is that *a posteriori*: in this case the optimizer carried out the *Pareto optimal set* and the DM chosen a solution ("search first and decide later"). Many researchers view these approaches as standard so that, in the most greater part of the circumstances, an MOOP is considered resolved once all *Pareto optimal solutions* are individualized. For instance, an extremely diffused *a posteriori approach* is denominated as *Aggregating* functions in which multiple objectives are combined into a single one. In this field, *Weighted Sum Method* is frequently adopted [28]: it consists of a single linear combination of individual objectives and a scalar parameter (so-called weighting coefficient) is used with different values in order to define the *Pareto front*. This method, as well as other Aggregating functions techniques, are not efficient for MOOPs because they are not able to find multiple solutions in a single run and multiple runs do not guarantee the definition of the true Pareto front [29]. Moreover, in the category of *a posteriori approaches*, *Evolutionary Multi-Objective Optimizations* are diffused. In Ref. [30] an algorithm for finding constrained Pareto optimal solutions based on the characteristics of a biological immune system (constrained multi-objective immune algorithm (CMOIA)) is proposed. In the field of EMOO, the most adopted algorithms are the multiple objective genetic algorithm (MOGA) [31], and the non-dominated sorting in genetic algorithm (NSGA) [32].

Particularly, in this work we adopt the NSGA-II, which is a new and modified version of the original NSGA method [33]. In Section 6, the principal characteristics of the algorithm for the resolution of the MOOP formulation are presented.

6. NSGA-II model for multiobjective optimization

NSGA-II is a diffused Pareto-based multiobjective evolutionary algorithm. In this section, we will analyze the fundamental aspects of the algorithm used for the resolution of MOOP formulated above. In order to use the decision variables directly, without coding, a real coded GA is used. The use of a real parameter is a powerful tool because it offers different advantages. In fact it is possible to make use of large domains for variables and to exploit the graduality of the functions with continuous variables [34]. The most important innovation introduced by NSGA-II is the sorting approach [33]. The population is sorted based on *rank* and *crowding distance*. In this way, it is possible to assign rank 1 for each individual of the first front where it finds non-dominated individuals. The second front is composed of the individuals (with rank 2) dominated by the individuals of the first front. The procedure is the same for each other front, with a progressive rank assigned. Behind, the algorithm calculates the crowding distance: this is a parameter that individualizes how close an individual is to its neighbours. Obviously, between two solutions with different non-domination ranks, the best point is that with a lower rank, but if both points belong to the same front we prefer the point that is located in a region with less number points. *Reproduction* (or *Selection*) is the first operator applied to population: particularly it is adopted to *Binary Tournament Selection*. It works as follows [35]: choose two individuals randomly from the population and copy the best individuals from this group into the intermediate population and finally repeat for all individuals in the population. In this case, selection is performed on non-domination ranks and crowding distance.

One of the most important operators in GA is *Crossover*. Crossover is a method for sharing information between strings. Like the reproduction operator, there are different crossover operators in the GA literature and generally the effectiveness of a method rather than another depends on the particular treated problem (i.e. coded/decode strategy, constraint/unconstraint optimization problem). For binary-coded GAs, a single-point crossover is frequently performed. With the aim to simulate the operation of a single-point binary crossover directly on real variables, the so-called *Simulated Binary Crossover* (SBX) is proposed in Ref. [36]. In this technique, the probability distribution used to create a child solution is derived to have a similar search power as that in a single-point crossover in binary-coded GAs [34]. It is given as follows:

$$P(\beta) = \begin{cases} \frac{1}{2}(\eta_c + 1)\beta_k^{\eta_c} & \text{if } 0 \leqslant \beta \leqslant 1, \\ \frac{1}{2}(\eta_c + 1)\frac{1}{\beta_k^{\eta_c+2}} & \text{otherwise.} \end{cases} \tag{50}$$

In Eq. (50), η_c is the distribution index for crossover operator and β_k is the so-called "spread factor" [36]. The procedure is the following: a random number $u_k \in [0,1]$ is generated using expression (50) and β_k is calculated with this formulation:

$$\beta_k = \begin{cases} (2u_k)^{1/(\eta_c+1)} & \text{if } u_k \leqslant 0.5, \\ \frac{1}{[2(1-u_k)]^{1/(\eta_c+1)}} & \text{otherwise.} \end{cases} \tag{51}$$

After obtaining β from Eq. (51), the children solution is calculated as follows:

$$c_{1,k} = \tfrac{1}{2}[(1 - \beta_k)p_{1,k} + (1 + \beta_k)p_{2,k}], \tag{52}$$

$$c_{2,k} = \tfrac{1}{2}[(1 + \beta_k)p_{1,k} + (1 - \beta_k)p_{2,k}]. \tag{53}$$

In Eqs. (52) and (53), $c_{i,k}$ is the ith child with the kth component, $p_{i,k}$ is the selected parent. It should be emphasized that with the aim to preserve some of the previously found good strings, not all strings in the population are taken into account in the crossover operation: for this reason, crossover is performed according to an assigned probability (crossover probability p_c).

Mutation is another adopted genetic operator: its role is to restore lost or unexpected genetic material into a population in order to prevent the premature convergence of GA: particularly, we adopt *polynomial mutation*. Mutation operator is performed on one string as follows [34]:

$$c_k = p_k + (p_k^u - p_k^l)\delta_k. \tag{54}$$

In Eq. (54), p_k is the parent with p_k^u and p_k^l are the upper bound and lower bound on the parent component, and c_k is the child. Mutation operator is based on δ_k, which is calculated from a polynomial distribution.

Table 1
NSGA-II algorithm for multiobjective optimization problem

1. Load data
- For GA
  - Population size and maximum generations
  - Crossover probability
  - Mutation probability
- For random vibrating structure
  - Input filter damping ratio and frequency (mean and variance)
  - Power spectral density
  - Main system frequency and damping ratio (mean and variance)
  - Tuned-main system mass ratio (mean and variance)

2. Initialize population
- Generate random population in the specified admissible domain
- Calculate OFs values
  - Solve continuous-time Lyapunov equations (by transformation of the matrices to the complex Schur form)

3. Sort the initialized population
- Sort the population using non-domination-sort. For each individual, rank and crowding distance are assigned

4. Loop for each generation
- Select the parents, which are fit for reproduction
  - Binary tournament selection based on the rank and crowding distance
- Genetic Operators on selected parents
  - Simulated binary crossover
  - Polynomial mutation
- The offspring population is combined with parents (size of intermediate population is double)
- Selection is performed to set the individuals of the next generation
  - Once the intermediate population is sorted, only the best individuals are selected based on its rank and crowding distance
- Create a new generation
  - Constant population size
- Close loop if stop criteria for max number of generation is verified, otherwise return on the top of loop

5. Report on results

First, a random number $r_k \in [0,1]$ is generated and δ_k is calculated with this formulation:

$$\delta_k = \begin{cases} (2r_k)^{1/(\eta_m+1)} - 1 & \text{if } r_k < 0.5, \\ 1 - [2(1-r_k)]^{1/(\eta_m+1)} & \text{if } r_k \geqslant 0.5. \end{cases} \tag{55}$$

In Eq. (55), η_m is the mutation distribution index. Also in this case, mutation operator is performed according to an assigned probability (mutation probability p_c).

Finally, in order to develop MOOP, the NSGA-II method whose scheme is shown in Table 1 has been adopted.

7. Numerical example

In order to solve the multiobjectives optimization problem proposed, several numerical applications have been carried out for specific levels of the main system and filter characteristics. These parameters,

Table 2
Mean and variation coefficient of system and filter characteristics

Input data		Value
Main system period (T_s)		0.45 s
Filter period (T_f)		0.35 s
Power spectral density (S_0)		1000 cm^2/s^3
Uncertain parameters d_i	Mean value $\mu(d_i)$	Variation coefficient $\rho(d_i)$
Main system parameters		
ω_S	13.95 rad/s	0.15
ξ_S	0.05	0.20
η	0.05	0.15
Filter parameters		
ω_f	18.62 rad/s	0.10
ξ_f	0.40	0.15

stochastically expressed by the mean and the variation coefficient, are considered to be deterministically known. The principal aim is to incorporate uncertainties in both the load and the structural model parameters.

All data with certain and uncertain parameters are listed in Table 2:

where

$$\text{the variation coefficient is :}\quad \rho_{\mathrm{TMD}} = \frac{\sigma_{\omega_{\mathrm{TMD}}}}{\mu_{\omega_S}} \tag{56}$$

$$\text{the frequency ratio is :}\quad \Psi = \frac{\mu_{\omega_S}}{\mu_{\omega_f}}. \tag{57}$$

A first analysis concerns the application of the conventional deterministic optimization method to obtain the OF surface. In Fig. 4, the mean (a) and the variance (b) of the OF are shown in the range of ρ_{TMD} and ξ_{TMD} (i.e. damping of the tuned ξ_T) investigated.

It can be noted that both the mean and the standard deviation of the OF present *extreme points* that are represented by a global minimum in the mean and by maximum in the variance trend. More in detail, the surface of the standard deviation shows not a unique but multiple picks very close to each other. Besides, it also can be noted that there is a quite perfect agreement between these observed pick points: for example, the global minimum of the mean seems to correspond to the region where maximum points take place in the surface of the standard deviation. More generally, it can be stated that when the mean of the OF increases, arising from the minimum point observed, the standard deviation tends to decrease according to the mean increasing. This consideration confirms the impossibility of achieving the perfect OF minimization both in terms of mean and standard deviation, because when it is possible to reduce the first one, the second one increases, having both of them counteracting effects with respect to the OF optimization. In fact, a more robust optimal solution can be obtained by improving structural performance both by reducing the OF mean value and by making more stable and less-sensitive the response to the uncertainty sources (i.e. reducing the OF standard deviation).

The above-drawn result show that the optimal solution, obtained by minimizing the expected value of the OF (the mean), is quite sensitive to the fluctuation of the uncertain parameters (as demonstrated by the corresponding high values of the standard deviation).

For this reason, a multiobjective robust design concept has been adopted to overcome this limitation and to provide more information about the structural optimization problem solution. The previous example has demonstrated that the satisfaction of the optimization problem regards two main aspects: the first concerns the necessity to satisfy critical performance requirements (minimizing the mean of the OF). The second one involves the need to maximize the robustness to uncertainty (minimizing the standard deviation of the OF).

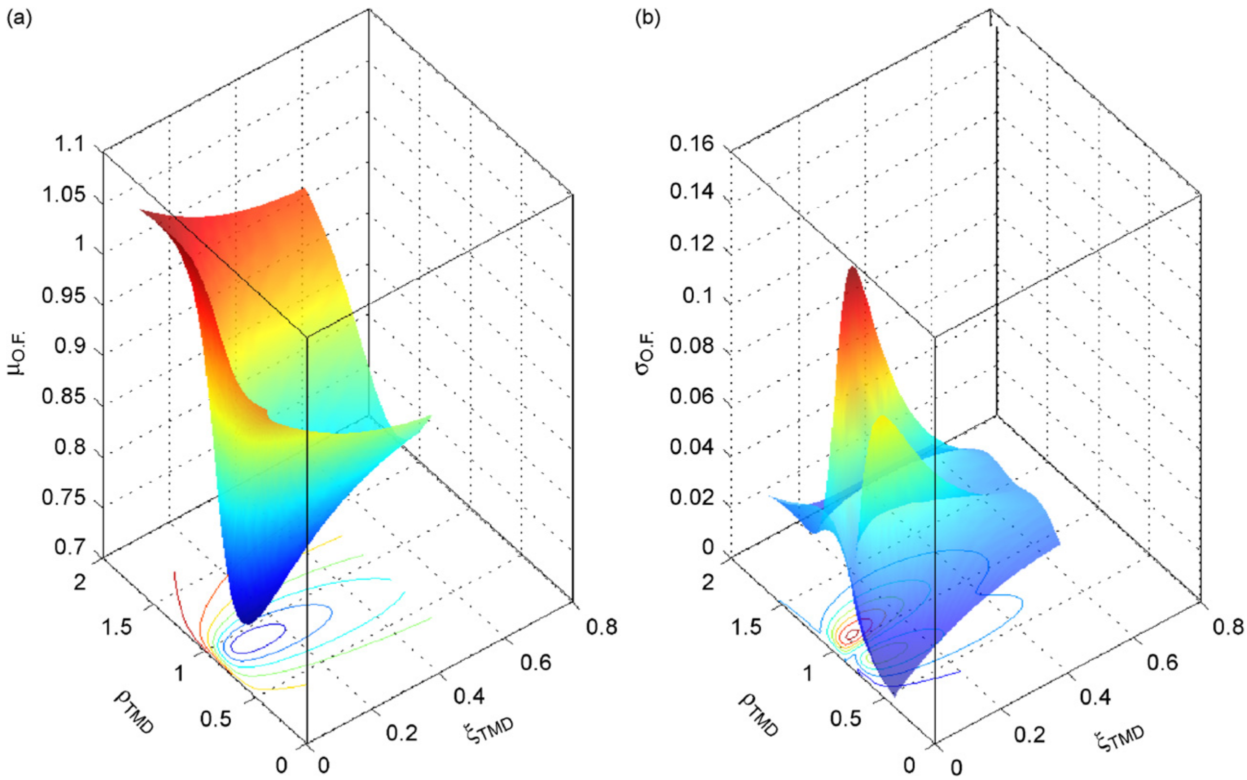

Fig. 4. Deterministic mean value (a) and standard deviation (b) of the OF for different values of ρ_{TMD} and ξ_{TMD}.

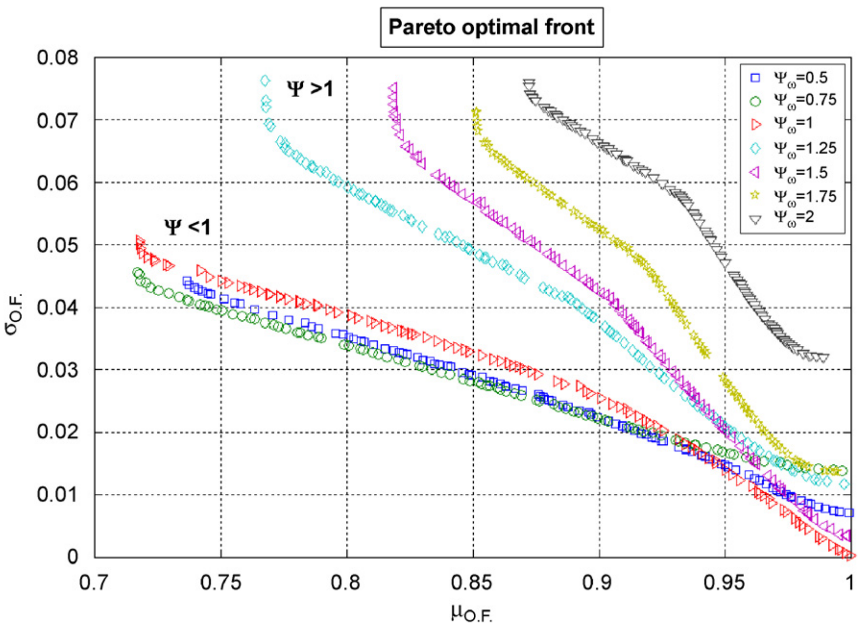

Fig. 5. Pareto optimal fronts for different frequency ratios.

Nevertheless, each criterion is in conflict with each other and for them it is not possible to define a universally approved criterion of ''optimum'' as in single objective optimization problem. Therefore, in MOOP, the aim is to produce a set of good compromised solutions, among which the decision maker can select. With this aim, a Pareto optimal set, for different uncertain configurations, has been plotted in Fig. 5 in a bi-dimensional Pareto domain (in mean μ_{OF} and standard deviation σ_{OF}) for different values of the frequency ratio ψ. In the figure, it can be seen that Pareto fronts show two main tendencies in function of the frequency ratio. These trend are described in the following.

- $\psi \leqslant 1$: Represents the situation in which the main system has a natural frequency smaller than that of the TMD. In this range, the fronts appear very close one to another. In detail, the set is characterized by lower values of the standard deviation, for a fixed mean value against the other ones. The maximum level of uncertainty achieved, in fact, is approximately at most the 50%.
- $\psi > 1$: Represents the situation in which the main system has a frequency higher than that of the TMD (i.e. condition that departs from the resonance). In this case the performance, expressed in terms of multiobjective structural problem optimization, gets worse. Pareto's fronts are characterized by larger values of the standard deviation at a fixed mean value; besides, it can be noted that each solution lies very far from another because of the more accented sensitivity of optimal points to the uncertainty parameters.

Another consideration concerns the shape of the fronts: they appear endowed with convexity, insofar they would not be determinable with conventional multiobjective optimization methods like, for example, the weighted sum method, because this last is a linear combination of the objectives. Instead the application of an evolutionary optimization problem approach in the Pareto's fronts definition is more appropriate.

Another group of graphs shows the distribution of the optimal points in a three-dimensional space of the mean (Fig. 6a) and the standard deviation (Fig. 6b) of the OF. In both graphs the presence of two very different tendencies, dependent by the frequency ratio, can be seen. For ψ less than unit, the distribution assumes a characteristic "handle" shape.

Better clearly, in Fig. 7, Pareto optimal solutions are plotted in the bi-dimensional DV domain (i.e. in terms of ρ_{TMD} and ξ_{TMD}). Even in this case, a markedly different trend is noticeable in function of the frequency ratio value. For $\psi \leqslant 1$, the optimal points are very close each other and follow a very precise way. This trend is restricted to the region of high values of ρ_{TMD} and of lower ones of ξ_{TMD}.

Once ψ increases, becoming larger than the unit, points appear very scattered and distributed in the whole domain, so it is no longer possible to recognize a well-defined run. It is also important to note that all optimal solutions start from the same point, but they differ because, in the under-resonance range under-resonance $\psi \leqslant 1$, points move horizontally toward the left, then bend and go down for smaller values of ξ_{TMD} and of ρ_{TMD}, describing a characteristic "handle" shape. Instead, in the over-resonance range $\psi > 1$, the front moves toward the right and initially follows a quite defined run, then scatters a lot, achieving higher values of the damping and lower ones of the variation coefficient.

A synthesis is reported in Table 3.

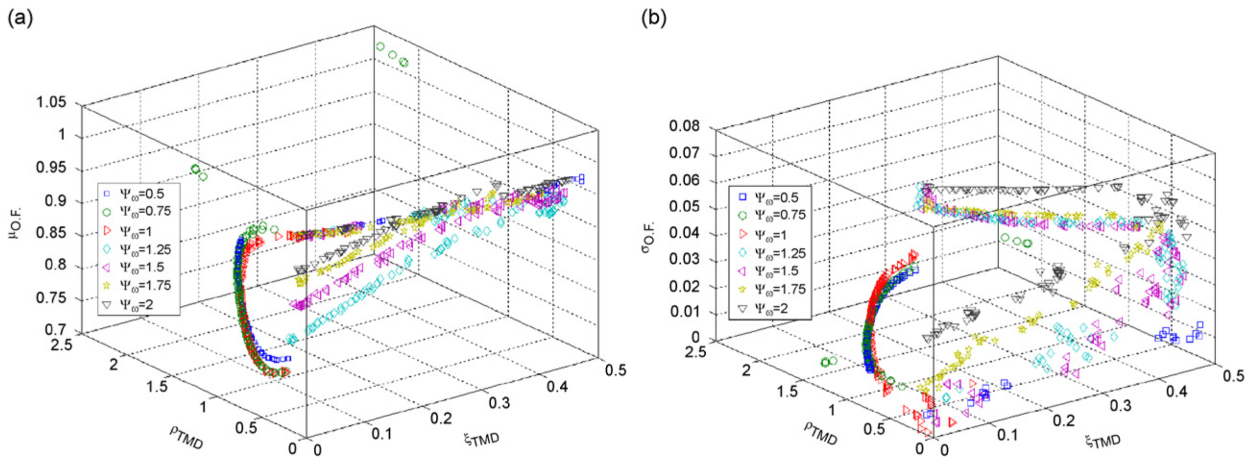

Fig. 6. Pareto optimal points in mean value (a) and standard deviation (b) of the OF for different values of ρ_{TMD} and ξ_{TMD}.

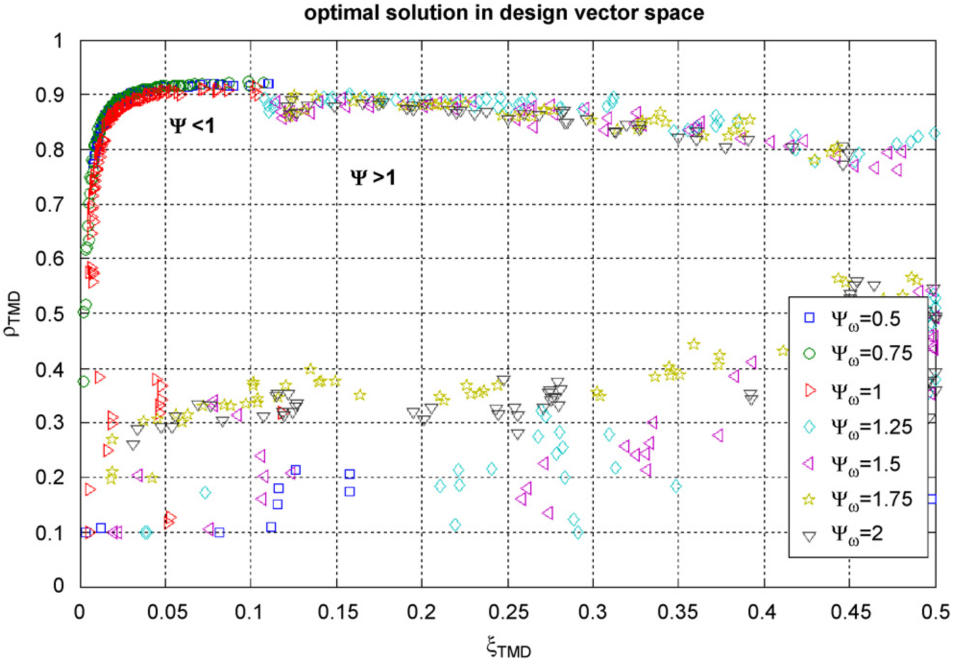

Fig. 7. Pareto optimal points in the design vector space.

Table 3
Main characteristics of optimal points distribution

Frequency ratio	Mean trend	Variance trend	Distribution in space state	Structural performance
$\psi \leqslant 1$	Well defined	Well defined	Primarily located in the range of low values of ξ_T	Low uncertainty at fixed mean value
$\psi > 1$	Quite scattered	Very scattered	Located in the range of high ξ_T values. Initially the front is well defined, then is scattered in the whole domain	High uncertainty at fixed mean value

Finally, Fig. 8 shows Pareto's optimal solutions plotted over the mean μ_{OF} (Fig. 8a) and the standard deviation σ_{OF} surface (Fig. 8b). In the first figure it can be seen that optimal solutions lies in correspondence of the global minimum point of the μ_{OF} surface and then they tend to get further from it as the uncertainty varies. The points go up in a well-defined trend that seems to follow the bending surface. Nevertheless, only in a very restricted region of the domain, characterised by larger values of ξ_{TMD}, the points appear more scattered.

Likewise, in Fig. 8b the optimal solution, are in correspondence of the maximum peaks and then go down by varying the mean value. Even in this case the Pareto set is well defined and all points are distributed very close each to another.

In the same figure, the optimal points and the contour lines of the OF surface are plotted overlapped in the same state space domain with the aim of observing more clearly the location of Pareto solutions with respect to the traditional optimization function, for frequency ratios. In the case of mean values distribution (column at left), the optimal points start from the global minimum of the OF mean surface; on the other side, in the graph of the standard deviation (column at right), the solutions go up from the middle region between the two peaks. In this representation, the different trend is even more evident: for frequency ratio less than unit, $\psi \leqslant 1$ (Fig. 8c and d), optimal points start, as stated before, from the extreme points and then move toward the left, both for the mean and for the standard deviation. On the contrary, in case of higher frequency ratio values $\psi \leqslant 1$ (Fig. 8e and f) the Pareto solutions move toward the right in correspondence of larger ξ_{TMD} values.

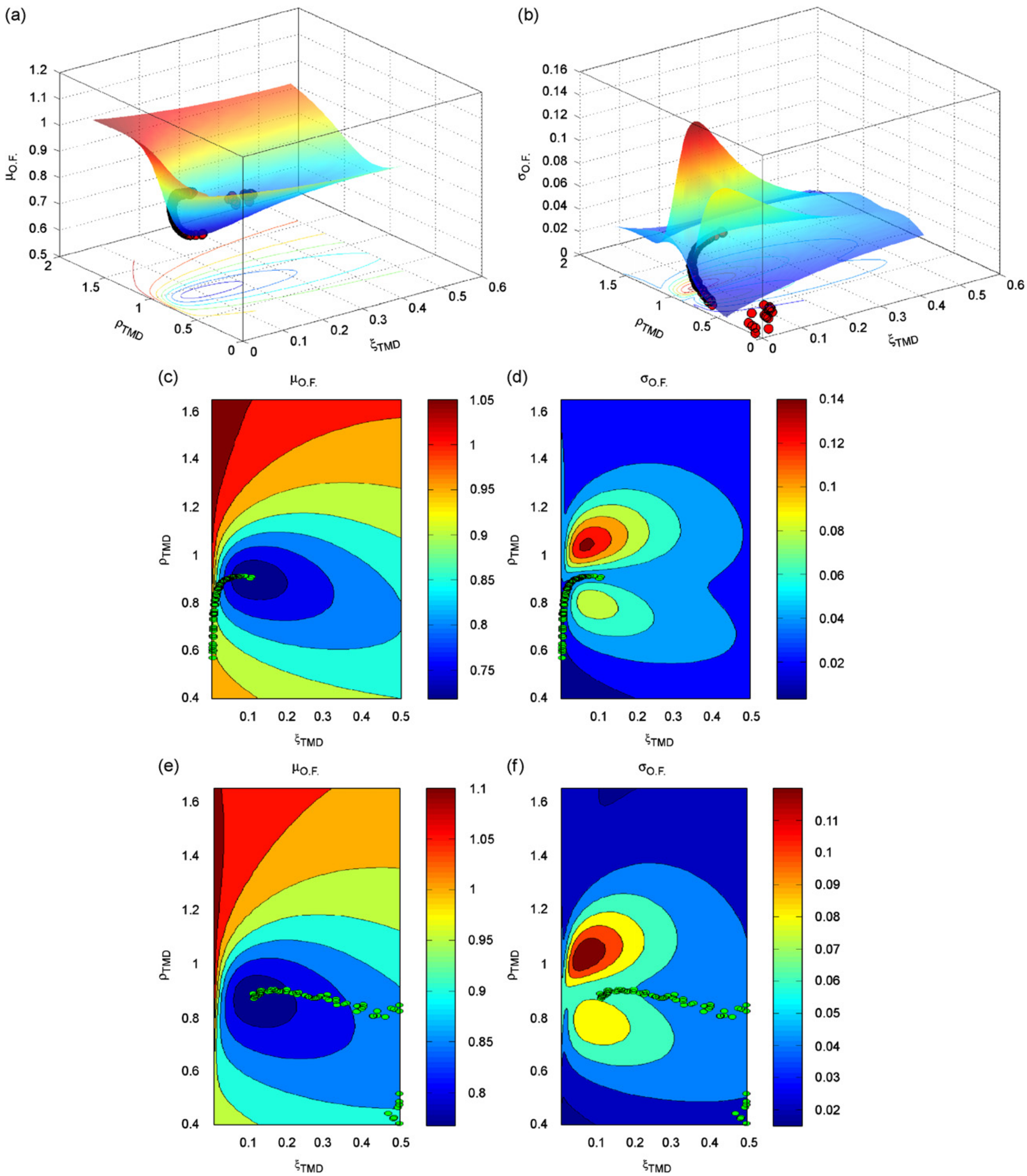

Fig. 8. OF mean value (first column) and standard deviation (second column) and Pareto solutions plotted in the space state domain of ρ_{TMD} and ξ_{TMD} (respectively, a and b) for different values of frequency ratio ψ in contour graph; $\psi\leqslant 1$ (c, d), $\psi>1$ (e, f).

These considerations are very important in the design chosen of the frequency ratio and the TMD characteristics, because the optimization structural performance expressed in terms of OF standard deviation minimization, is very sensitive to these parameters variation.

8. Summary and conclusions

A robust optimal design criterion for a single TMD device in case of random vibrations is proposed. This vibration control problem refers to the case of systems subjected to dynamic actions having stochastic nature that could be modelled by a stochastic process. Robustness is obtained by finding solutions that take into account not only the absolute performance but also considering its sensitivity to system parameters variation due to uncertainty. The dynamic input is represented by a random base acceleration, modelled by a stationary filtered white noise process, in order to take into account loads–structures resonant effects. The main system is described by a sdof system; it is assumed that structural stiffness and damping, tuned mass ratio, filter main frequency and damping are affected by uncertainty in their evaluation. In detail, the parameters are described by a mean value and a variance (standard deviation), assuming that they are all mutually statistically independent. No other information is considered in order to assume the given probability densities function. The OF definition is here assumed as main structure covariance displacement. To perform the robust optimum design, its mean and standard deviations are numerically evaluated. Robustness is formulated as a MOOP, in which both the mean and the standard deviation of the deterministic OF are minimized. The results show a significant improvement in performance control and in limitation of the OF dispersion, in comparison with standard conventional solutions. In detail, some interesting conclusions could be done with reference to the results obtained for the adopted examples. With reference to TMD effectiveness in vibration reduction, the real structural performance obtained by using conventional optimization has a reduced effectiveness with respect to those obtained when system parameters uncertainty is properly considered. With reference to the obtained robust solutions it can be noted that they are able to control and limit the OF dispersion by limiting its standard deviation. Moreover, this goal is achieved by finding optimal solutions in terms of DV that induces an increase of OF mean value. The application of the Pareto concept, to research the solution of the MOOP, is able to evaluate the optimal choice of the DV, that represents a compromise solution to guarantee acceptable level relative displacement. An Evolutionary approach by means of a GA has been used to solve the MOOP and to search the population of non-inferior solutions. Numerical examples show that all assessments and information drawn by means of this kind of a computational model, cannot be obtained by the use of the simple conventional optimization technique. The following conclusions can be made:

- Robust TMD optimal solutions are obtained by varying the *damping ratio* by a conventional deterministic optimization method. This result is independent of the input frequency content. Increasing of $\xi_{\mathrm{TMD}}^{\mathrm{opt}}$ increases with uncertainty.
- Robust TMD optimal solutions are obtained by varying the *TMD frequency*. With reference to this parameter, the required variation is function of the input frequency content. For frequency ratio $\psi < 1$, the TMD frequency decreases, meanwhile it must increase if $\psi \geqslant 1$.

The proposed method could also be used when more accurate information about uncertain parameters are known, for instance by using different probability distributions, as the beta one. Besides, the number of uncertain sources could be incremented to take into account some other parameters, without a serious computational cost increment.

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